THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2010F Advanced Calculus I
Homework 2

Due Date: 11:59pm, 27 June, 2025

1. Let $A \subset \mathbb{R}^n$ be closed and bounded and let $f : A \to \mathbb{R}$ be a continuous function on A. Show that $f(A)$ must be a bounded set in $\mathbb{R}$.

2. Prove the theorem of Lagrange multipliers:

   Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be open and let $f, g : \Omega \to \mathbb{R}$ be C^1 functions on Ω. Let $S = g^{-1}(c) = \{x \in \Omega : g(x) = c\}$ be the level set of g with respect to the value $c \in \mathbb{R}$. Suppose*

   (a) $a \in S$ is a local extremum of f restricted to S;

   (b) $\nabla g(a) \neq 0$.

   Then
   $$\begin{cases} \nabla f(a) = \lambda \nabla g(a) & \text{for some } \lambda \in \mathbb{R} \\ g(a) = c. \end{cases}$$

3. Let $c \in \mathbb{R}$ and let $g(x, y) = Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F$ be a conic section where A, B, C, D, E, F are constants and A, B, C are all not zero. Recall the special types of conic sections below:

   (a) Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a, b > 0$

   (b) Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, a, b > 0$

   (c) Parabola: $y = ax^2, a \neq 0$

   (d) Degenerate cases $(a, b > 0)$:

      i. A point $(0, 0)$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$

      ii. Empty set: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$

      iii. A pair of intersecting lines: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \Rightarrow \frac{x}{a} = \pm \frac{y}{b}$

      iv. A pair of lines $x^2 = c \Rightarrow x = \pm\sqrt{c}$ which is a pair of parallel lines if $c > 0$, a "double" line if $c = 0$ and the empty set if $c < 0$.

   Prove the following fact: By a change of coordinates, any quadratic constraint $g(x, y) = c$ can be transformed to one of the forms listed above.